• Own calculators can be used in the exam.
• You may take the examination paper with you.

1. Are the following statements true or false? (No justification is needed. Correct answer: 1 p, incorrect: $-\frac{1}{2}$ p, no answer: 0 p.) The total number of points is rounded up to the nearest integer.

   (a) Stability of a filter is checked by determining whether the zeros of the transfer function are within the unit circle.

   (b) Two parallel LTI systems can always be implemented as a single system.

   (c) Linearity of the phase response ensures that all the frequencies in the signal are delayed the same amount in seconds.

   (d) The sampling rate is converted to 1.5 times the original by first decimating by 2 and then interpolating by 3.

   (e) Aliasing is prevented in the A/D conversion by setting the sampling rate at least the same as the maximum frequency in the analog signal.

   (f) IIR filters are always stable.

2. (a) Calculate the DFT of the vector $x(n) = (3, 3, -1, 2)^T$. (1p)

   (b) What is the DFT matrix in the case $N = 2$? (2p)

   (c) An LTI system with transfer function $H(z)$ is said to be minimum-phase if
      - the system is stable and
      - it has an inverse system (transfer function $\frac{1}{H(z)}$), which is also stable.

      Both systems are assumed to be causal. Is the system
      \[ H(z) = \frac{(z^2 - 1.2)(z^2 - 0.9)}{z^4 - z^3 + 0.125z^2} \]

      minimum-phase? Justify your answer. (3p)

3. The transfer function of an LTI system is:

   \[ H(z) = \frac{-2z^2 + 2z - 1}{z^2 + 4}. \]

   (a) Determine the difference equation between the input $x(n)$ and the output $y(n)$. The asked equation is thus of the form (2p)

   \[ b_0 y(n) + b_1 y(n - 1) + b_2 y(n - 2) = a_0 x(n) + a_1 x(n - 1) + a_2 x(n - 2). \]

   (b) Draw the pole-zero plot. (2p)
(c) Is the system stable? Why/why not? (2p)

4. Design using the window design method a high-pass filter (i.e. find out its impulse response) satisfying the following requirements:

<table>
<thead>
<tr>
<th>Requirement</th>
<th>Specifications</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stopband</td>
<td>[0 kHz, 4 kHz]</td>
</tr>
<tr>
<td>Passband</td>
<td>[6 kHz, 15 kHz]</td>
</tr>
<tr>
<td>Passband ripple</td>
<td>0.04 dB</td>
</tr>
<tr>
<td>Minimum stopband attenuation</td>
<td>24 dB</td>
</tr>
<tr>
<td>Sampling frequency</td>
<td>30 kHz</td>
</tr>
</tbody>
</table>

Use the tables below. (6p)

5. (a) Complete the block diagram below so that it represents first-order noise shaping. (3p)

(b) The figure below shows training data that has two classes: "red squares" (■) and "blue plus signs" (⊕). Which of the two classes the black point (4.6, 3.7) marked in the figure belongs to when you use the

   i. 1-NN classifier,
   ii. 3-NN classifier,
   iii. 5-NN classifier?

Justify your answers. (3p)
### Tables

<table>
<thead>
<tr>
<th>Filter type</th>
<th>Impulse response when $n \neq 0$: $2f_c \text{sinc}(n \cdot 2\pi f_c)$</th>
<th>Impulse response when $n = 0$: $2f_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low-pass</td>
<td>$-2f_c \text{sinc}(n \cdot 2\pi f_c)$</td>
<td></td>
</tr>
<tr>
<td>High-pass</td>
<td>$2f_2 \text{sinc}(n \cdot 2\pi f_2) - 2f_1 \text{sinc}(n \cdot 2\pi f_1)$</td>
<td>$1 - 2f_c$</td>
</tr>
<tr>
<td>Band-pass</td>
<td>$2f_1 \text{sinc}(n \cdot 2\pi f_1) - 2f_2 \text{sinc}(n \cdot 2\pi f_2)$</td>
<td>$2(f_2 - f_1)$</td>
</tr>
</tbody>
</table>

| Name of the window function | Transition bandwidth (normalized) | Passband ripple (dB) | Minimum stopband attenuation | Window expression $w(n)$, when $|n| \leq (N - 1)/2$ |
|-----------------------------|----------------------------------|----------------------|-----------------------------|-------------------------------------------------|
| Rectangular                 | 0.9/N                            | 0.7416               | 21 dB                       | $1$                                             |
| Bartlett                    | 3.05/N                           | 0.4752               | 25 dB                       | $1 - \frac{2|n|}{N - 1}$                        |
| Hanning                     | 3.1/N                            | 0.0546               | 44 dB                       | $0.5 + 0.5 \cos \left(\frac{2\pi n}{N}\right)$ |
| Hamming                     | 3.3/N                            | 0.0194               | 53 dB                       | $0.54 + 0.46 \cos \left(\frac{2\pi n}{N}\right)$ |
| Blackman                    | 5.5/N                            | 0.0017               | 74 dB                       | $0.42 + 0.5 \cos \left(\frac{2\pi n}{N}\right) + 0.08 \cos \left(\frac{4\pi n}{N}\right)$ |

$$ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Some Wikipedia pages that might be useful

Suppose two classes of observations have means $\mu_0$, $\mu_1$, and covariances $\Sigma_0$, $\Sigma_1$. Then the linear combination of features $\mathbf{w} \cdot \mathbf{z}$ will have means $\mathbf{w} \cdot \mu_i$ and variances $\mathbf{w}^T \Sigma_i \mathbf{w}$ for $i = 0, 1$. Fisher defined the separation between these two distributions to be the ratio of the variance between the classes to the variance within the classes:

$$S = \frac{\sigma^2_{\text{between}}}{\sigma^2_{\text{within}}} = \frac{(\mathbf{w} \cdot \mu_1 - \mathbf{w} \cdot \mu_0)^2}{\mathbf{w}^T \Sigma_1 \mathbf{w} + \mathbf{w}^T \Sigma_0 \mathbf{w}} = \frac{\mathbf{w}^T (\Sigma_0 - \Sigma_1) \mathbf{w}}{\mathbf{w}^T (\Sigma_0 + \Sigma_1) \mathbf{w}}$$

This measure is, in some sense, a measure of the signal-to-noise ratio for the class labelling. It can be shown that the maximum separation occurs when

$$\mathbf{w} \propto (\Sigma_0 + \Sigma_1)^{-1} (\mu_1 - \mu_0)$$

When the assumptions of LDA are satisfied, the above equation is equivalent to LDA.

Be sure to note that the vector $\mathbf{w}$ is the normal to the discriminant hyperplane. As an example, in a two dimensional problem, the line that best divides the two groups is perpendicular to $\mathbf{w}$.

Generally, the data points to be discriminated are projected onto $\mathbf{w}$, then the threshold that best separates the data is chosen from analysis of the one-dimensional distribution. There is no general rule for the threshold. However, if projections of points from both classes exhibit approximately the same distributions, a good choice would be the hyperplane between projections of the two means, $\mathbf{w} \cdot \mu_0$ and $\mathbf{w} \cdot \mu_1$. In this case the parameter $c$ in threshold condition $\mathbf{w} \cdot \mathbf{z} > c$ can be found explicitly:

$$c = \mathbf{w} \cdot \frac{1}{2} (\mu_0 + \mu_1) = \frac{1}{2} \mu_0 \Sigma_1^{-1} \mu_1 = \frac{1}{2} \mu_0 \Sigma_0^{-1} \mu_0.$$
A more condensed form of the difference equation is:

\[ y[n] = \frac{1}{a_0} \left( \sum_{i=0}^{P} b_i x[n-i] - \sum_{j=1}^{Q} a_j y[n-j] \right) \]

which, when rearranged, becomes:

\[ \sum_{j=0}^{Q} a_j y[n-j] = \sum_{i=0}^{P} b_i x[n-i] \]

To find the transfer function of the filter, we first take the Z-transform of each side of the above equation, where we use the time-shift property to obtain:

\[ \sum_{j=0}^{Q} a_j z^{-j} Y(z) = \sum_{i=0}^{P} b_i z^{-i} X(z) \]

We define the transfer function to be:

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{P} b_i z^{-i}}{\sum_{j=0}^{Q} a_j z^{-j}} \]

Considering that in most IIR filter designs coefficient \( a_0 \) is 1, the IIR filter transfer function takes the more traditional form:

\[ H(z) = \frac{\sum_{i=0}^{P} b_i z^{-i}}{1 + \sum_{j=1}^{Q} a_j z^{-j}} \]

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**Inversion of 2 × 2 matrices**

The cofactor equation listed above yields the following result for 2 × 2 matrices. Inversion of these matrices can be done as follows:[8]

\[
A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]

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**Techniques**

Conceptual approaches to sample-rate conversion include: converting to an analog continuous signal, then re-sampling at the new rate, or calculating the values of the new samples directly from the old samples. The latter approach is more satisfactory, since it introduces less noise and distortion.[5] Two possible implementation methods are as follows:

1. If the ratio of the two sample rates is (or can be approximated by)\^[8] [9] a fixed rational number \(L/M\), generate an intermediate signal by inserting \(L - 1\) 0s between each of the original samples. Low-pass filter this signal at half of the lower of the two rates. Select every \(M\)-th sample from the filtered output, to obtain the result.[5]

2. Treat the samples as geometric points and create any needed new points by interpolation. Choosing an interpolation method is a trade-off between implementation complexity and conversion quality (according to application requirements). Commonly used are: ZOH (for film/video frames), cubic (for image processing) and windowed sinc function (for audio).